Start at 10:35

1. Show that

$$\frac{d}{dz}\log z = \frac{1}{z} \qquad (|z|>0,\ \alpha < \arg z < \alpha + 2\pi)$$

$$\alpha \in \mathbb{R}.$$

2 Find and sketch with orientations, the images of the hyperbolas

① $x^2 - y^2 = C_1 \quad (C_1 < 0)$

② $2xy = C_2 \quad (C_2 > 0)$

under $w = z^2$.

$z_0 \neq 0$

$z_0 = x + iy$ or $z_0 = re^{i\theta}$

$\Rightarrow \quad x = r\cos\theta, \quad y = r\sin\theta$

$f(z) =: w = u(x,y) + i\,v(x,y)$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

Multi-variable Chain Rule

Let $x = x(t)$, $y = y(t)$ be differentiable at t and suppose that $z = f(x,y)$ is diff. at $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is diff at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$u_x\cos\theta + u_x\sin\theta$

$\Rightarrow u_r = u_x\cos\theta + u_y\sin\theta, \quad u_\theta = -u_x r\sin\theta + u_y r\cos\theta$

$v_r = v_x\cos\theta + v_y\sin\theta, \quad v_\theta = -v_x r\sin\theta + v_y r\cos\theta$

Since $\log z$ is diff. for $|z|>0$,

$\alpha < \arg z < \alpha + 2\pi$.

$\Rightarrow u_x = v_y, \quad v_x = -u_x$ at z_0.

$v_r = -u_y\cos\theta + u_x\sin\theta, \quad v_\theta = u_y r\sin\theta + u_x r\cos\theta$

$\Rightarrow r u_r = v_\theta, \quad u_\theta = -r v_r$ at z_0.

THM. (Page 69)

If $f'(z_0)$ exists, $z_0 = r_0 e^{i\theta_0}$, then

$$f'(z_0) = e^{-i\theta}(u_r + i v_r)\Big|_{r=r_0,\ \theta=\theta_0}.$$

Pf:
$$\begin{aligned} f'(z) &= u_x + i v_x \\ &= \left(\cos\theta\, u_r - \tfrac{1}{r}\sin\theta \cdot u_\theta\right) \\ &\quad + i\left(\cos\theta\, v_r - \tfrac{1}{r}\sin\theta\, v_\theta\right) \\ &\overset{CR}{=} (\cos\theta\, u_r + \sin\theta\, v_r) + i(\cos\theta\, v_r - \sin\theta\, u_r) \\ &= (\cos\theta - i\sin\theta)u_r + i(\cos\theta - i\sin\theta)v_r \\ &= e^{-i\theta}(u_r + i v_r) \end{aligned}$$

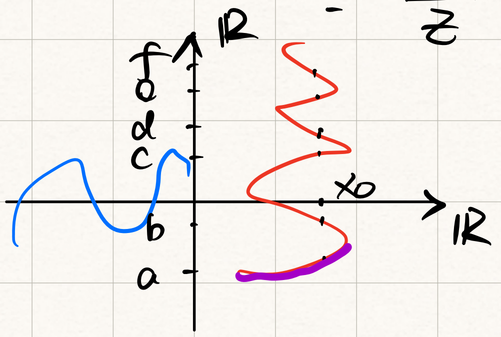
For $\log z = \underbrace{\ln r}_{u} + \underbrace{i\theta}_{iv} \quad (r>0,\ \alpha<\theta<\alpha+2\pi)$.

$u(r,\theta) = \ln r, \quad v(r,\theta) = \theta$.

$u_r = \frac{1}{r}, \quad u_\theta = 0, \quad v_r = 0, \quad v_\theta = 1$

Check $\begin{cases} u_r, u_\theta, v_r, v_\theta \text{ are cont. at } \forall z = re^{i\theta} \\ ru_r = v_\theta, \quad u_\theta = -rv_r \end{cases}$

$$\Rightarrow \frac{d}{dz}\log z = e^{-i\theta}(u_r + iv_r)$$
$$= e^{-i\theta}\left(\frac{1}{r} + i\cdot 0\right)$$
$$= \frac{1}{re^{i\theta}}$$
$$= \frac{1}{z} \qquad (|z|>0,\ \alpha < \arg z < \alpha + 2\pi).$$



$f: X \to Y$

$f: X \to 2^Y$ Set-valued fct.

$f(x_0) = \{a, b, c, d, e, f\}$.

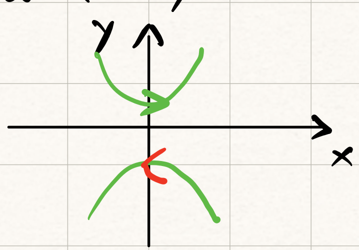
2. $w = u + iv$, $\quad f(z) = z^2 = w$

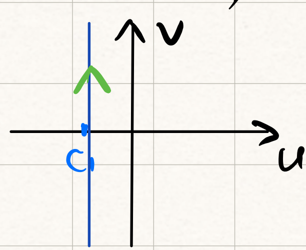
$z = x + iy$,

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

① $u = x^2 - y^2 = c_1$, $\quad c_1 < 0$, $\quad v = 2xy$





Consider $y > 0$

$y = \sqrt{x^2 - C_1} > 0$

$v = 2x\sqrt{x^2 - C_1} \quad (-\infty < x < +\infty)$

When $x \nearrow$, then $v \nearrow$.

Consider $y < 0$

$y = -\sqrt{x^2 - C_1} < 0$

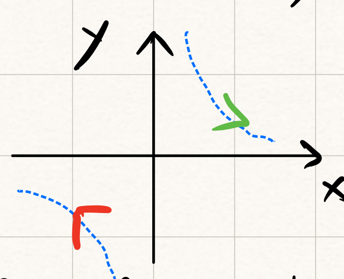
$v = -2x\sqrt{x^2 - C_1} \quad (-\infty < x < +\infty)$

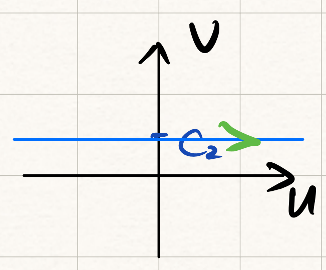
When $x \nearrow$, then $v \downarrow$. ✗

$x \downarrow$, $v \nearrow$ ✓

② $v = 2xy = C_2 > 0 \qquad y = \frac{C_2}{2x}$

$u = x^2 - y^2$.





Consider 1st quadrant : $y = \frac{C_2}{2x}$ $(x > 0)$

then $u = x^2 - \frac{C_2^2}{4x^2} \quad (0 < x < \infty)$

$\Rightarrow u \nearrow$ as $x \nearrow$

Consider 3rd quadrant. $y = \frac{C_2}{2x}$ $(x < 0)$

$u = x^2 - \frac{C_2^2}{4x^2} \quad (-\infty < x < 0)$

$\Rightarrow$ $u \uparrow$ as $x \downarrow$ ✓

$u \downarrow$ as $x \uparrow$ ✗